三角恒等式

$$\begin{cases} \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \\ \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \\ \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \\ \tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\cdot\tan y} \end{cases}$$

$$\sin\alpha\sin\beta=\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$$

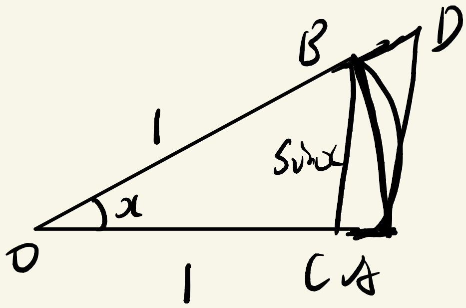
$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$\begin{cases} \sin 2x=2\sin x\cos x \\ \cos 2x=\cos^2 x-\sin^2 x=1-2\sin^2 x=2\cos^2 x-1 \\ \tan 2x=\dfrac{2\tan x}{1-\tan^2 x} \end{cases}$$

$$\begin{cases} \cos^2 x+\sin^2 x=1 \\ \tan^2 x+1=\sec^2 x \end{cases} \qquad \cot^2 x+1=\csc^2 x$$

几个重要极限

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$



$$S_{\triangle OAB}<S_{\frown OAB}<S_{\triangle OAD} \Rightarrow$$

$$\frac{1}{2}\sin x<\frac{1}{2}x<\frac{1}{2}\tan x \Rightarrow$$

$$\sin x<x<\frac{\sin x}{\cos x}$$

$OB=OA=1$

$CB=\sin x$

$AD=\tan x$

(1) 当 $x\to 0^+$

$$\sin x<x \Rightarrow \frac{\sin x}{x}<1 \qquad x<\frac{\sin x}{\cos x} \Rightarrow \frac{\sin x}{x}>\cos x$$

故 $\cos x < \frac{\sin x}{x} < 1$ $\lim\limits_{x\to 0^+}\cos x = \lim\limits_{x\to 0^+} 1 = 1$，故

$$\lim_{x\to 0^+}\frac{\sin x}{x} = 1$$

(2) 当 $x\to 0^-$

$\sin x < x \Rightarrow \frac{\sin x}{x} > 1$

$x < \frac{\sin x}{\cos x} \Rightarrow \frac{\sin x}{x} < \cos x$，故 $1 < \frac{\sin x}{x} < \cos x$

$\lim\limits_{x\to 0^-}\cos x = \lim\limits_{x\to 0^-} 1 = 1$，故

$$\lim_{x\to 0^-}\frac{\sin x}{x} = 1$$

综上，可知 $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ 证毕 #.

$\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$

不证

常用等价无穷小

$\sin x \sim x \quad (x\to 0)$

上已证

$\tan x \sim x \quad (x\to 0)$

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x} = 1\cdot 1 = 1$$

$1-\cos x \sim \frac{1}{2}x^2 \quad (x \to 0)$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\lim_{x\to 0}\frac{1}{1+\cos x}$$

$$= 1\cdot\frac{1}{2} = \frac{1}{2}$$

$\arcsin x \sim x \quad (x\to 0)$

令 $\arcsin x = t \Rightarrow x = \sin t \quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\lim_{x\to 0}\frac{\arcsin x}{x} = \lim_{t\to 0}\frac{t}{\sin t} = 1$$

$\arctan x \sim x \quad (x\to 0)$

令 $\arctan x = t \Rightarrow x = \tan t \quad t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\lim_{x\to 0}\frac{\arctan x}{x} = \lim_{t\to 0}\frac{t}{\tan t} = 1$$

$\sqrt[n]{1+x} - 1 \sim \frac{1}{n}x \quad (x\to 0)$

不证.

$\ln(1+x) \sim x \quad (x\to 0)$

$$\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \lim_{x\to 0}\log_a(1+x)^{\frac{1}{x}} = \log_a \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \log_a e$$

$$= \frac{\ln e}{\ln a} = \frac{1}{\ln a}$$

当 $a=e$ 时，即有 $\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$

$e^x-1\sim x\ (x\to 0)$

$\lim\limits_{x\to 0}\frac{a^x-1}{x}$，令 $a^x-1=t$，则 $x=\log_a^{t+1}$

$\lim\limits_{x\to 0}\frac{a^x-1}{x}=\lim\limits_{t\to 0}\frac{t}{\log_a^{t+1}}=\ln^a$

当 $a=e$ 时，即有 $\lim\limits_{x\to 0}\frac{e^x-1}{x}=1$

$(1+x)^\alpha-1\sim\alpha x\quad(x\to 0)\ (\alpha\in R)$

$\lim\limits_{x\to 0}\frac{(1+x)^\alpha-1}{x}$，令 $(1+x)^\alpha-1=t$

$\lim\limits_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\lim\limits_{x\to 0}\frac{(1+x)^\alpha-1}{\ln(1+x)^\alpha}\cdot\frac{\alpha\cdot\ln(1+x)}{x}=$

$\lim\limits_{t\to 0}\frac{t}{\ln^{t+1}}\cdot\lim\limits_{x\to 0}\frac{\alpha\cdot\ln(1+x)}{x}=1\cdot\alpha=\alpha$

综上：

$\sin x\sim x$，$\tan x\sim x$，$\arcsin x\sim x$，$\arctan x\sim x$，

$1-\cos x\sim\frac{1}{2}x^2$

$\sqrt[n]{1+x}-1\sim\frac{1}{n}x$，$(1+x)^\alpha-1\sim\alpha x\ (\alpha\in R)$

$\ln^{1+x}\sim x$，$\log_a^{1+x}\sim\frac{x}{\ln a}$；$e^x-1\sim x$，$a^x-1\sim x\ln^a$

（以上 x 都趋于 0 时成立）

常用求导公式

$C' = 0$

$(x^{\mu})' = \mu \cdot x^{\mu-1} \ (\mu \in R)$

$(a^x)' = a^x \cdot \ln a \ (a>0 且 a \neq 1)$

$$(a^x)' = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} a^x \cdot \frac{a^{\Delta x} - 1}{\Delta x} = a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

据上述等价无穷小之关系可知：$\lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = \ln a$

因此，$(a^x)' = a^x \cdot \ln a$

$(e^x)' = e^x$

$(\log_a x)' = \frac{1}{x \ln a}$

$$(\log_a x)' = \lim_{\Delta x \to 0} \frac{\log_a^{x+\Delta x} - \log_a^{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\log_a^{1+\frac{\Delta x}{x}}}{\Delta x} = \log_a \lim_{\Delta x \to 0} \left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}$$

令 $\frac{\Delta x}{x} = t$，$\Rightarrow \frac{1}{\Delta x} = \frac{1}{tx} = \frac{1}{t} \cdot \frac{1}{x}$

则上式 $= \log_a \lim_{t \to 0} (1+t)^{\frac{1}{t} \cdot \frac{1}{x}} = \log_a e^{\frac{1}{x}} = \frac{1}{x} \log_a e = \frac{1}{x} \cdot \frac{1}{\ln a}$

$(\ln x)' = \frac{1}{x}$

$(\sin x)' = \cos x$

不证.

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

$(\sec x)' = \sec x \tan x$

$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{0 - (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \tan x \cdot \sec x.$

$(\cot x)' = -\csc^2 x$

同上不证.

$(\csc x)' = -\csc x \cot x$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$y = \arcsin x \quad x = \sin y \quad y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y' = (\arcsin x)' = \frac{1}{\sin y'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

$(\arctan x)' = \frac{1}{1+x^2}$

$y = \arctan x \quad x = \tan y \quad y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$y' = (\arctan x)' = \frac{1}{\tan' y} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

同上不证.

$(\text{arccot}\, y)' = -\frac{1}{1+x^2}$

综上:

$c' = 0$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\sec x)' = \sec x \tan x$

$(x^\mu)' = \mu x^{\mu-1}$

$(a^x)' = a^x \ln a$

$(e^x)' = e^x$

$(\log_a x)' = \frac{1}{x \ln a}$

$(\ln x)' = \frac{1}{x}$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

$(\arctan x)' = \frac{1}{1+x^2}$

$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$ $(\cot x)' = -\csc^2 x$

$(\csc x)' = -\csc x \cot x$

基本积分表

(1) $\int k\,dx = kx + C \quad k \in R$

(2) $\int x^{\mu}\,dx = \frac{x^{\mu+1}}{\mu+1} + C \ (\mu \neq -1)$

(3) $\int \frac{1}{x}\,dx = \ln|x| + C$

(4) $\int \frac{1}{1+x^2}\,dx = \arctan x + C$

(5) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$

$\cancel{(6) \int -\frac{1}{1+x^2}\,dx = \operatorname{arccot} x + C}$

(7) $\int \frac{1}{\cos^2 x}\,dx = \int \sec^2 x\,dx = \tan x + C$

(8) $\int \frac{1}{\sin^2 x}\,dx = \int \csc^2 x\,dx = -\cot x + C$

(9) $\int \cos x\,dx = \sin x + C$

(10) $\int \sin x\,dx = -\cos x + C$

(11) $\int \sec x \tan x\,dx = \sec x + C$

(12) $\int \csc x \cot x\,dx = -\csc x + C$

(13) $\int e^x\,dx = e^x + C$

(14) $\int a^x\,dx = \frac{a^x}{\ln a} + C$

常用积分式

(1) $\int \frac{1}{a^2+x^2}dx = \int \frac{1}{a^2}\frac{1}{1+(\frac{x}{a})^2}dx = \int \frac{1}{a}\frac{1}{1+(\frac{x}{a})^2}d\frac{x}{a}$

$= \frac{1}{a}\arctan\frac{x}{a} + C$

(2) $\int \frac{1}{x^2-a^2}dx \ (a\neq 0) = \frac{1}{2a}\int \left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$

$= \frac{1}{2a}\left[\int \frac{1}{x-a}d(x-a) - \int \frac{1}{x+a}d(x+a)\right]$

$= \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

(3) $\int \sec x\,dx = \int \frac{1}{\cos x}dx = \int \frac{\cos x}{\cos^2 x}dx = \int \frac{d\sin x}{1-\sin^2 x}$

令 $\sin x = u$，原式 $= \int \frac{du}{1-u^2} = \frac{1}{2}\int \frac{1}{1+u}du + \frac{1}{2}\int \frac{1}{1-u}du$

$= \frac{1}{2}\left[\int \frac{1}{1+u}d(1+u) - \int \frac{1}{1-u}d(1-u)\right]$

$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| = \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} = \frac{1}{2}\ln\frac{(1+\sin x)^2}{1-\sin^2 x}$

$= \ln\left|\frac{1+\sin x}{\cos x}\right| + C = \ln|\sec x + \tan x| + C.$

(4) $\int \sqrt{a^2-x^2}\,dx \ (a>0)$

令 $x = a\sin t \quad t\in[-\frac{\pi}{2}, \frac{\pi}{2}]$

$(\cos 2t =$

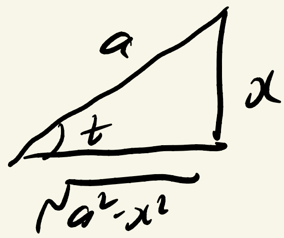
$\int \sqrt{a^2-x^2}\,dx = \int \sqrt{a^2-a^2\sin^2 t}\,d\,a\sin t = a^2\int \cos^2 t\,dt = a^2\int \frac{1+\cos 2t}{2}dt$

$= a^2\left(\frac{1}{2}t + \frac{1}{4}\sin 2t\right) + C = a^2\left(\frac{1}{2}t + \frac{1}{2}\sin t\cos t\right) + C$

(4) $t = \arcsin \frac{x}{a}$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{a^2}{2} \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} + C$$

$$= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \cdot \sqrt{a^2 - x^2} + C$$



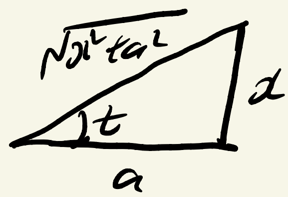
(5) $\int \frac{1}{\sqrt{x^2 + a^2}} dx \quad (a > 0)$

令 $x = a \tan t \quad t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \int \frac{1}{\sqrt{a^2 \tan^2 t + a^2}} d\, a \tan t = \int \frac{a \sec^2 t}{a \sec t} dt = \int \sec t\, dt$$

$$= \ln |\sec t + \tan t| + C$$

$$= \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| + C$$



$$= \ln \left| \sqrt{x^2 + a^2} + x \right| + C_1 \quad (C_1 = C - \ln a)$$

(6) $\int \frac{1}{\sqrt{x^2 - a^2}} dx \quad (a > 0)$

令 $x = a \sec t$，由于 $x \in (-\infty, -a) \cup (a, +\infty)$，因此分情况讨论.

当: (1) $x > a$ 时，$\Rightarrow \sec t > 1 \Rightarrow t \in (0, \frac{\pi}{2})$

(2) $x < -a$ 时，$\Rightarrow \sec t < -1 \Rightarrow t \in (\pi, \frac{3\pi}{2})$

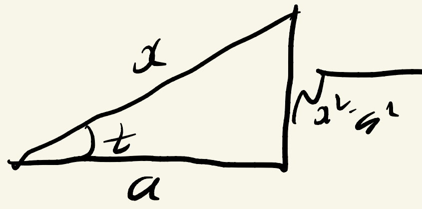
(之所以取 $(\pi, \frac{3\pi}{2})$ 是因为在该区间内 $\tan t > 0$)

(1) $$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{1}{\sqrt{a^2 \sec^2 t - a^2}} d\, a \sec t = \int \frac{a \sec t \tan t\, dt}{a \tan t} = \int \sec t\, dt$$

$= \ln|\sec t + \tan t| + C$

$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$

$= \ln|x + \sqrt{x^2-a^2}| + C_1 \quad (C_1 = C - \ln a)$



(2) 同上

(7) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{a\,d\frac{x}{a}}{a\sqrt{1-\left(\frac{x}{a}\right)^2}} = \int \frac{d\frac{x}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \arcsin\frac{x}{a} + C$$

(8) $\int \sqrt{x^2-a^2}\,dx$

$$\int \sqrt{x^2-a^2}\,dx = \int 1\cdot \underset{u}{\sqrt{x^2-a^2}}\,\underset{v}{dx} = x\cdot\sqrt{x^2-a^2} - \int x\cdot d\sqrt{x^2-a^2}$$

(分部积分)

$$= x\cdot\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\cdot\sqrt{x^2-a^2} - \int \frac{x^2-a^2}{\sqrt{x^2-a^2}}\,dx - \int \frac{a^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\cdot\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}\,dx - a^2\int\frac{1}{\sqrt{x^2-a^2}}\,dx \Rightarrow$$

$$2\int\sqrt{x^2-a^2}\,dx = x\cdot\sqrt{x^2-a^2} - a^2\int\frac{1}{\sqrt{x^2-a^2}}\,dx$$

$$= x\cdot\sqrt{x^2-a^2} - a^2\ln|x+\sqrt{x^2-a^2}| + C \Rightarrow$$

$$\int\sqrt{x^2-a^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2-a^2} - a^2\ln|x+\sqrt{x^2-a^2}|\right) + C$$

(9) $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2+a^2} + a^2\ln\left|x+\sqrt{x^2+a^2}\right|\right) + C$

引法同上不证.

综上 **基本积分表2.**

(15) $\int \tan x\,dx = -\ln|\cos x| + C$

(16) $\int \cot x\,dx = \ln|\sin x| + C$

(17) $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

(18) $\int \csc x\,dx = \ln|\csc x - \cot x| + C$

(19) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + C$

(20) $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left|x+\sqrt{x^2+a^2}\right| + C$

(21) $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x+\sqrt{x^2-a^2}\right| + C$

(22) $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

(23) $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$

(24) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$

(25) $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2+a^2} + a^2\ln\left|x+\sqrt{x^2+a^2}\right|\right) + C$

(26) $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2-a^2} - a^2\ln\left|x+\sqrt{x^2-a^2}\right|\right) + C$